

HALL ALGEBRAS ASSOCIATED TO TRIANGULATED CATEGORIES

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Dedicated to Yanan Lin on the occasion of his 50th birthday

ABSTRACT. By counting with triangles and the octohedral axiom, we find a direct way to prove the formula of Toën in [13] for a triangulated category with (left) homological-finite condition.

1. INTRODUCTION

Let k be the finite field \mathbb{F}_q with q elements and A be a finite dimensional k -algebra. Ringel associated the module category of A an associative algebra $\mathcal{H}(A)$, which now is called Ringel-Hall algebra, and used it to give a realization of the positive part of simple Lie algebra when A is a hereditary algebra of finite representation type (see [10] and [11]). In general, the idea of Ringel-Hall algebra constructs an associative algebra $\mathcal{H}(\mathcal{A})$ from an abelian category \mathcal{A} . The isomorphism classes of object in \mathcal{A} generate the vector space $\mathcal{H}(\mathcal{A})$ with multiplication $[X] * [Y] = \sum_{[L]} F_{XY}^L [L]$, where F_{XY}^L is called Hall number and is the number of subobject L' of L such that $L' \cong X, L/L' \cong Y$. Moreover, Ringel and Green showed when A is an arbitrary hereditary algebra, the composition subalgebra of $\mathcal{H}(A)$ gives a realization of the positive part of the quantum enveloping algebra of the corresponding Kac-Moody algebras (see [9] and [1], also [12]). So the next question is, asked by Ringel in [9], to recover the whole Lie algebra and the whole quantized enveloping algebra. A direct idea is to use Drinfeld Double to piece together two Borel parts as showed in [14]. However, this construction is not “intrinsic”, i.e., not naturally induced by the module category of A . Therefore, one need extend the module category of A to a larger category.

To deal with this question, several important developments have been made. One is to use the 2-period triangulated category to define an analog multiplication of Hall multiplication (see [8]). Although this multiplication is not associative in general, the Lie bracket induced by it satisfies Jacobi identity and a geometric method is verified feasible to define this Lie bracket directly over the complex field (see [15]). On the other hand, Kapranov in [3] defined Heisenberg doubles for hereditary categories and attached an associative algebra to the derived category of any hereditary category. Recently, Toën made a remarkable development in this direction. He defined an associative algebra corresponding to a dg category by

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using model categories and fibre product of model categories. A key formula for the derived Hall numbers, i.e. structure constant of the multiplication analogous to Hall number, is given (see Proposition 5.1 in [13] or Section 3 in the following). In Remark 5.3 in [13], Toën asked whether one can define the derived Hall algebra of any triangulated category under some finiteness condition by his formula as the structure constant. The purpose of this note is to find a direct method to deduce Toën's formula for arbitrary triangulated category under some finiteness conditions (see section 2 for these conditions). We combine the methods in [8] and in [13]. Our idea is as follows. First, we recognize one of the main reasons that the multiplication defined in [8] does not satisfy the associativity is that the action of $\text{Aut } X \times \text{Aut } Y$ on $W(X, Y; L)$ is not free (see Section 2 for the definitions of these and the following notations). So we naturally consider the replacement $V(X, Y; L)$ by $|\text{Hom}(L, Y)_{X[1]}|/|\text{Aut } Y|$ or by $|\text{Hom}(X, L)_Y|/|\text{Aut } X|$. Proposition 2.5' shows the explicit relation between $V(X, Y; L)$ and $|\text{Hom}(L, Y)_{X[1]}|/|\text{Aut } Y|$ or $|\text{Hom}(X, L)_Y|/|\text{Aut } X|$. However, it is not proper to set $|\text{Hom}(L, Y)_{X[1]}|/|\text{Aut } Y|$ or $|\text{Hom}(X, L)_Y|/|\text{Aut } X|$ as the derived Hall numbers since this definition is not symmetric, i.e., $|\text{Hom}(L, Y)_{X[1]}|/|\text{Aut } Y| \neq |\text{Hom}(X, L)_Y|/|\text{Aut } X|$. In fact, Proposition 2.5' implies a symmetric expression. A simple computation shows that the proof of the associativity comes down to confirming the symmetry of the other expression (see Definition 3.1 and Proposition 3.4), and the symmetry of the latter heavily depends on the octahedral axiom.

Finally we should pay attention to the following points. One of the next tasks is to construct Toën's formula for a 2-period orbit category. For the enveloping algebra \mathcal{U} of a simple split Lie algebra of type ADE. An arbitrarily large finite dimensional quotient of \mathcal{U} can be constructed in terms of constructible functions on a "triple variety" by Lusztig (see [4]) or in terms of the homology of a "triple variety" by Nakajima (see [5]). It will be very interesting to look for the relations between the construction by "triple variety" and the Toën's formula for a 2-period orbit category.

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2. CALCULATION WITH TRIANGLES

Given a finite field k with q elements, let \mathcal{C} be a k -additive triangulated category with the translation $T = [1]$. We always assume in this paper that (1) the homomorphism space $\text{Hom}_{\mathcal{C}}(X, Y)$ for any two objects X and Y in \mathcal{C} is a finite dimensional k -space, and (2) the endomorphism ring $\text{End } X$ for any indecomposable object X is finite dimensional local k -algebra. We note that the above two conditions imply the Krull-Schmidt theorem holds in \mathcal{C} , i.e., any object in \mathcal{C} can be decomposed into the direct sum of finitely many indecomposable objects. Moreover, we always assume that \mathcal{C} is (left) locally homological finite, i.e., $\sum_{i \geq 0} \dim_k \text{Hom}(X[i], Y) < \infty$ for any X and Y in \mathcal{C} . We will use fg to denote the composition of morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, and $|A|$ the cardinality of a finite set A . For example, the bounded derived category $\mathcal{D}^b(A)$ of the module category $\text{mod } A$ of a finite dimensional k -algebra A satisfies all conditions as above. However, its 2-period orbit category

$\mathcal{D}^b(A)/T^2$, which is a triangulated category if A is hereditary, does not satisfy the homological finiteness property.

The following is an easy result in [8].

Lemma 2.1. *Given a triangle of form*

$$(1) \quad M \xrightarrow{\begin{pmatrix} f_1 & f_2 \end{pmatrix}} N_1 \oplus N_2 \xrightarrow{\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}} L \xrightarrow{h} M[1]$$

If $f_2 = 0$, then it is isomorphic to the triangle of form

$$(2) \quad M \xrightarrow{\begin{pmatrix} f_1 & f_2 \end{pmatrix}} N_1 \oplus N_2 \xrightarrow{\begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}} L_1 \oplus L_2 \xrightarrow{\begin{pmatrix} h_1 \\ 0 \end{pmatrix}} M[1]$$

where $g_{22} : N_2 \rightarrow L_2$ is an isomorphism and

$$(3) \quad M \xrightarrow{f_1} N_1 \xrightarrow{g_{11}} L_1 \xrightarrow{h_1} M[1]$$

is a triangle.

The lemma shows the triangle (1) is the direct sum of (3) and the following contractible triangle (also see [6]):

$$0 \longrightarrow N_2 \xrightarrow{\sim} L_2 \longrightarrow 0$$

Now we recall some notation in [8].

Given any object X, Y, Z, L, L' and M in \mathcal{C} , we define

$$W(X, Y; L) = \{(f, g, h) \in \text{Hom}(X, L) \times \text{Hom}(L, Y) \times (Y, X[1]) \mid \\ X \xrightarrow{f} L \xrightarrow{g} Y \xrightarrow{h} X[1] \text{ is a triangle}\}$$

The following we simply write (f, g, h) is a triangle to denote $X \xrightarrow{f} L \xrightarrow{g} Y \xrightarrow{h} X[1]$ is a triangle.

The action of $\text{Aut } X \times \text{Aut } Y$ on $W(X, Y; L)$ induces the orbit space

$$V(X, Y; L) = \{(f, g, h)^\wedge \mid (f, g, h) \in W(X, Y; L)\}$$

where

$$(f, g, h)^\wedge = \{(af, gc^{-1}, ch(a[1])^{-1}) \mid (a, c) \in \text{Aut } X \times \text{Aut } Y\}$$

For any $((f, g, h)(l, m, n)) \in W(X, Y; L) \times W(Z, L; M)$, we define

$$((f, g, h), (l, m, n))^* \\ = \{((fb^{-1}, bgc^{-1}, ch), (dl, mb^{-1}, bn(d[1])^{-1})) \mid b \in \text{Aut } L, c \in \text{Aut } Y, d \in \text{Aut } Z\}$$

So we have the orbit space

$$(W(X, Y; L) \times W(Z, L; M))^* \\ = \{((f, g, h), (l, m, n))^* \mid ((f, g, h), (l, m, n)) \in W(X, Y; L) \times W(Z, L; M)\}$$

Dually, for any $((l', m', n'), (f', g', h')) \in W(Z, X; L') \times W(L', Y; M)$ we define

$$((l', m', n'), (f', g', h'))^* \\ = \{((dl'b'^{-1}, b'm', n'(d[1])^{-1}), (b'f', g'c^{-1}, ch'(b'[1])^{-1})) \mid d \in \text{Aut } Z, b' \in \text{Aut } L', c \in \text{Aut } Y\}$$

Then the orbit space is

$$(W(Z, X; L') \times W(L', Y; M))^* \\ = \{((l', m', n'), (f', g', h'))^* \mid ((l', m', n'), (f', g', h')) \in W(Z, X; L') \times W(L', Y; M)\}$$

We define the action of $\text{Aut } Z$ on $W(Z, L; M)$ as follows: For any $(l, m, n) \in W(Z, L; M)$, $d(l, m, n) = (dl, m, n(d[1])^{-1})$ for any $(l, m, n) \in W(Z, L; M)$ and any $d \in \text{Aut } Z$. We denote the orbit by

$$(l, m, n)_Z^* = \{(dl, m, n(d[1])^{-1}) \mid d \in \text{Aut } Z\}$$

then the orbit space is

$$W(Z, L; M)_Z^* = \{(l, m, n)_Z^* \mid (l, m, n) \in W(Z, L; M)\}.$$

Dually, we also have the action of $\text{Aut } L$ on $W(Z, L; M)$.

$$(l, m, n)_L^* = \{(l, mb^{-1}, bn) \mid b \in \text{Aut } L\}$$

and

$$W(Z, L; M)_L^* = \{(l, m, n)_L^* \mid (l, m, n) \in W(Z, L; M)\}.$$

We have the following diagram to help understanding (see [8]).

$$(4) \quad \begin{array}{ccccccc} Z & \xlongequal{\quad} & Z & & & & \\ \downarrow l' & & \downarrow l & & & & \\ L' & \xrightarrow{\quad f' \quad} & M & \xrightarrow{\quad g' \quad} & Y & \xrightarrow{\quad h' \quad} & L'[1] \\ \downarrow m' & & \downarrow m & & \parallel & & \downarrow m'[1] \\ X & \xrightarrow{\quad f \quad} & L & \xrightarrow{\quad g \quad} & Y & \xrightarrow{\quad h \quad} & X[1] \\ \downarrow n' & & \downarrow n & & & & \\ Z[1] & \xlongequal{\quad} & Z[1] & & & & \end{array}$$

Let $X, Y \in \mathcal{C}$. Define $\text{rad Hom}(X, Y)$, the radical of $\text{Hom}_{\mathcal{C}}(X, Y)$, to be

$$\text{rad Hom}(X, Y) = \{f \in \text{Hom}_{\mathcal{C}}(X, Y) \mid gfh \text{ is not an isomorphism}$$

for any $g : A \rightarrow X$ and $h : Y \rightarrow A$ with indecomposable $A\}$

Lemma 2.2. *Let $X, Y \in \mathcal{C}$ and $n \in \text{Hom}_{\mathcal{C}}(X, Y)$, then there exists the decompositions $X = X_1(n) \oplus X_2(n)$, $Y = Y_1(n) \oplus Y_2(n)$ and $a \in \text{Aut } X$, $c \in \text{Aut } Y$ such that $anc = \begin{pmatrix} n'_{11} & 0 \\ 0 & n'_{22} \end{pmatrix}$, where n'_{11} is an isomorphism between $X_1(n)$ and $Y_1(n)$, $n'_{22} \in \text{rad Hom}(X_2(n), Y_2(n))$.*

Proof. Let $X = \bigoplus_i X_i$ and $Y = \bigoplus_j Y_j$ be the direct sums of indecomposable objects. For any indecomposable summands X_i of X and Y_j of Y , the morphism n induces the morphism $n_{ij} \in \text{Hom}(X_i, Y_j)$. If $n_{ij} \in \text{rad Hom}(X_i, Y_j)$ for all i, j , then $n \in \text{rad Hom}(X, Y)$. So we only need to take $X_1 = Y_1 = 0$. If there exist some isomorphism n_{ij} , we may assume it is n_{11} without loss of generality, then we have

$$X = X_1 \oplus X' \xrightarrow{\begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix}} Y = Y_1 \oplus Y'$$

Consider $\begin{pmatrix} 1 & 0 \\ -n_{21}n_{11}^{-1} & 1 \end{pmatrix} \in \text{Aut } X$ and $\begin{pmatrix} 1 & -n_{11}^{-1}n_{12} \\ 0 & 1 \end{pmatrix} \in \text{Aut } Y$, then

$$\begin{pmatrix} 1 & 0 \\ -n_{21}n_{11}^{-1} & 1 \end{pmatrix} \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} \begin{pmatrix} 1 & -n_{11}^{-1}n_{12} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} n_{11} & 0 \\ 0 & n'_{22} \end{pmatrix}.$$

The similar discussion works for n'_{22} . By induction, we achieve the claim of the lemma. \square

Remark 2.3. Any $(l, m, n)^\wedge \in V(Z, L; M)$ has the representative of the form:

$$(5) \quad Z \xrightarrow{\begin{pmatrix} 0 \\ l_2 \end{pmatrix}} M \xrightarrow{\begin{pmatrix} 0 & m_2 \end{pmatrix}} L \xrightarrow{\begin{pmatrix} n_{11} & 0 \\ 0 & n_{22} \end{pmatrix}} Z[1]$$

where $Z = Z_1 \oplus Z_2$, $L = L_1 \oplus L_2$ and n_{11} is an isomorphism between L_1 and $Z_1[1]$, $n_{22} \in \text{radHom}(L_2, Z_2[1])$. Depending on Lemma 1.1, it can be decomposed into the direct sum of two triangles, in which one is a contractible triangle.

In order to simplify the notation, for $X, Y \in \mathcal{C}$, we set

$$\{X, Y\} = \prod_{i>0} |\text{Hom}(X[i], Y)|^{(-1)^i}.$$

Lemma 2.4. For any $\alpha = (l, m, n) \in W(Z, L; M)$, set

$$n\text{Hom}(Z[1], L) = \{b \in \text{End} L \mid b = nt \text{ for some } t \in \text{Hom}(Z[1], L)\}$$

and

$$\text{Hom}(Z[1], L)n = \{d \in \text{End} Z[1] \mid d = sn \text{ for some } s \in \text{Hom}(Z[1], L)\}$$

We have

$$(1) \quad |n\text{Hom}(Z[1], L)| = \prod_{i>0} \frac{|\text{Hom}(M[i], L)|^{(-1)^i}}{|\text{Hom}(Z[i], L)|^{(-1)^i} |\text{Hom}(L[i], L)|^{(-1)^i}} = \frac{\{M, L\}}{\{Z, L\}\{L, L\}}$$

$$(2) \quad |\text{Hom}(Z[1], L)n| = \prod_{i>0} \frac{|\text{Hom}(Z[i], M)|^{(-1)^i}}{|\text{Hom}(Z[i], L)|^{(-1)^i} |\text{Hom}(Z[i], Z)|^{(-1)^i}} = \frac{\{Z, M\}}{\{Z, L\}\{Z, Z\}}$$

Proof. We only prove the first identity. It is similar to prove the second. Applying $\text{Hom}(-, L)$ to the triangle $Z \xrightarrow{l} M \xrightarrow{m} L \xrightarrow{n} Z[1]$ we get the long exact sequence

$$\cdots \rightarrow \text{Hom}(L[1], L) \rightarrow \text{Hom}(M[1], L) \rightarrow \text{Hom}(Z[1], L) \xrightarrow{n^*} \text{Hom}(L, L) \rightarrow \cdots$$

Since $n\text{Hom}(Z[1], L) = \text{Image of } n^*$, we have the identity in this lemma. \square

By Lemma 1.2, for any $L \xrightarrow{n} Z[1]$, there exist the decompositions $L = L_1(n) \oplus L_2(n)$, $Z[1] = Z_1[1](n) \oplus Z_2[1](n)$ and $b \in \text{Aut } L$, $d \in \text{Aut } Z$ such that $bn(d[1])^{-1} = \begin{pmatrix} n_{11} & 0 \\ 0 & n_{22} \end{pmatrix}$ and the induced maps $n_{11} : L_1(n) \rightarrow Z_1[1](n)$ is an isomorphism and $n_{22} : L_2(n) \rightarrow Z_2[1](n)$ contains no isomorphism component. The above decomposition only depends on the equivalence class of n up to an isomorphism. Let $\alpha = (l, m, n)^\wedge \in V(Z, L; M)$, the classes of α and n are determined to each other in $V(Z, L; M)$. We may denote n by $n(\alpha)$ and $L_1(n)$ by $L_1(\alpha)$ respectively.

The following is a refinement of Lemma 7.1 in [8].

Proposition 2.5. We have

$$|W(Z, L; M)_Z^*| = \sum_{\alpha \in V(Z, L; M)} \frac{|\text{Aut } L| |\text{End } L_1(\alpha)|}{|n(\alpha)\text{Hom}(Z[1], L)| |\text{Aut } L_1(\alpha)|}$$

Proof. Consider the map

$$\varphi : W(Z, L; M)_Z^* \rightarrow V(Z, L; M)$$

sending $(l, m, n)^*$ to $(l, m, n)^\wedge$. The action of $\text{Aut } L$ on $W(Z, L; M)$ naturally induces the action on $W(Z, L; M)_Z^*$ with the orbit space $V(Z, L; M)$. Hence, for any $(l, m, n)^\wedge \in V(Z, L; M)$, $\varphi^{-1}((l, m, n)^\wedge)$ is just the orbit of $(l, m, n)^*$ under the action of $\text{Aut } L$. Let $\alpha = (l, m, n)^\wedge$. For $(l, m, n)^*$, we denote its stable subgroup by $G((l, m, n)^*)$. Let $n' = bn(d[1])^{-1} = \begin{pmatrix} n_{11} & 0 \\ 0 & n_{22} \end{pmatrix}$, $l' = dl$ and $m' = mb^{-1}$ where $b \in \text{Aut } L$ and $d \in \text{Aut } Z$. Then

$$|G((l, m, n)^*)| = |G((l', m', n')^*)|$$

$$G((l', m', n')^*) = \{b \in \text{Aut } L \mid (l', m'b^{-1}, bn')^* = (l', m', n')^*\}$$

By definition, we have $G((l', m', n')^*) = \{b \in \text{Aut } L \mid m'b = m'\}$, i.e.

$$1 - G((l', m', n')^*) = \{b' \in \text{End } L \mid m'b' = 0 \text{ and } 1 - b' \in \text{Aut } L\}$$

On the other hand, $m'b' = 0$ if and only if $b' = n't$ for some $t \in \text{Hom}(Z[1], L)$. Let $t = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$ with respect to the above decompositions. Then we need

$$b = \begin{pmatrix} 1 - n_{11}t_{11} & -n_{11}t_{12} \\ -n_{22}t_{21} & 1 - n_{22}t_{22} \end{pmatrix} \in \text{Aut } L$$

The morphism $n_{22} \in \text{rad Hom}(L_2, Z_2[1])$ implies $n_{22}t_{21}$ is nilpotent and $1 - n_{22}t_{22} \in \text{Aut } L_2$. Hence, $b \in \text{Aut } L$ if and only if $1 - n_{11}t_{11} \in \text{Aut } L_1$. We obtain

$$\begin{aligned} & |G((l', m', n')^*)| \\ &= |\text{Aut } L_1(\alpha)| |n_{11} \text{Hom}(Z_1[1](\alpha), L_2(\alpha))| |n_{22} \text{Hom}(Z_2[1](\alpha), L_1(\alpha))| |n_{22} \text{Hom}(Z_2[1], L_2)| \\ &= |n \text{Hom}(Z[1], L)| |\text{Aut } L_1(\alpha)| / |\text{End } L_1(\alpha)| \end{aligned}$$

On the other hand, we have $|n \text{Hom}(Z[1], L)| = |n' \text{Hom}(Z[1], L)|$ by the above remark. We complete the proof of the proposition. \square

Dually, we also have

$$|W(Z, L; M)_L^*| = \sum_{\alpha \in V(Z, L; M)} \frac{|\text{Aut } Z| |\text{End } Z_1(\alpha)|}{|\text{Hom}(Z, L[-1])n(\alpha)[-1]| |\text{Aut } Z_1(\alpha)|}$$

where $Z_1(\alpha) \simeq L_1(\alpha)[-1]$.

As in [13], we denote by $\text{Hom}(Z, M)_L$ the subset of $\text{Hom}(Z, M)$ consisting of morphisms $l : Z \rightarrow M$ whose cone $\text{Cone}(l)$ is isomorphic to L . There is a natural action of the group $\text{Aut } Z$ on $\text{Hom}(Z, M)_L$ by $d \cdot l = dl$, the orbit is denoted by l^* and the orbit space is denoted by $\text{Hom}(Z, M)_L^*$. Dually We also have the subset $\text{Hom}(M, L)_{Z[1]}$ of $\text{Hom}(M, L)$ with the group action of $\text{Aut } L$ and the orbit space $\text{Hom}(M, L)_{Z[1]}^*$.

Proposition 2.6. *There exist bijections:*

$$W(Z, L; M)_Z^* \rightarrow \text{Hom}(M, L)_{Z[1]} \text{ and } W(Z, L; M)_L^* \rightarrow \text{Hom}(Z, M)_L.$$

Moreover, $|V(Z, L; M)| = |\text{Hom}(M, L)_{Z[1]}^*| = |\text{Hom}(Z, M)_L^*|$.

Proof. We have the natural surjections:

$$W(Z, L; M)_Z^* \xleftarrow{\pi_1} W(Z, L; M) \xrightarrow{\pi_2} \text{Hom}(M, L)_{Z[1]}$$

$\pi_1^{-1}((l, m, n)^*) = \{(dl, m, n(d[1])^{-1}) \mid d \in \text{Aut } Z\}$ and $\pi_2^{-1}(m) = \{(l, m, n) \mid (l, m, n) \text{ is a triangle}\}$. It is clear that $\pi_1^{-1}((l, m, n)^*) \subseteq \pi_2^{-1}(m)$. On the other hand, for any $(l_1, m, n_1), (l_2, m, n_2) \in \pi_2^{-1}(m)$, We have the following diagram:

$$\begin{array}{ccccccc} Z & \xrightarrow{l_1} & M & \xrightarrow{m} & L & \xrightarrow{n_1} & Z[1] \\ \uparrow d & & \parallel & & \parallel & & \\ Z & \xrightarrow{l_2} & M & \xrightarrow{m} & L & \xrightarrow{n_2} & Z[1] \end{array}$$

Using (Tr3) in the triangulated category axioms, there exists an isomorphism $d \in \text{Aut } Z$ such that the above diagram commutative. So $(l_2, m, n_2) = (dl_1, m, n_1(d[1])^{-1})$. This shows $\pi_1^{-1}((l, m, n)^*) = \pi_2^{-1}(m)$. Hence, we naturally define a bijection: $W(Z, L; M)_Z^* \rightarrow \text{Hom}(M, L)_{Z[1]}$ sending $(l, m, n)^*$ to m . Dually, we also have a bijection: $W(Z, L; M)_L^* \rightarrow \text{Hom}(Z, M)_L$. Consider the maps:

$$\varphi : W(Z, L; M)_Z^* \rightarrow V(Z, L; M)$$

sending $(l, m, n)^*$ to $(l, m, n)^\wedge$ and

$$\phi : \text{Hom}(M, L)_{Z[1]} \rightarrow \text{Hom}(M, L)_{Z[1]}^*$$

sending m to m^* . Then

$$\varphi^{-1}((l, m, n)^\wedge) = \{(l, mb^{-1}, bn)^* \mid b \in \text{Aut } L\}$$

and

$$\phi^{-1}(m^*) = \{mb^{-1} \mid b \in \text{Aut } L\}$$

the bijection between $W(Z, L; M)_Z^*$ and $\text{Hom}(M, L)_{Z[1]}$ induces the bijection between $\varphi^{-1}((l, m, n)^\wedge)$ and $\phi^{-1}(m^*)$. This shows $|V(Z, L; M)| = |\text{Hom}(M, L)_{Z[1]}^*|$. \square

By Proposition 2.6, Proposition 2.5 and its dual formula can be rewritten as follows.

Proposition 2.5' The following equalities hold.

$$\frac{|\text{Hom}(M, L)_{Z[1]}|}{|\text{Aut } L|} \cdot \frac{\{M, L\}}{\{Z, L\} \cdot \{L, L\}} = \sum_{\alpha \in V(Z, L; M)} \frac{|\text{End } L_1(\alpha)|}{|\text{Aut } L_1(\alpha)|}$$

$$\frac{|\text{Hom}(Z, M)_L|}{|\text{Aut } Z|} \cdot \frac{\{Z, M\}}{\{Z, L\} \cdot \{Z, Z\}} = \sum_{\alpha \in V(Z, L; M)} \frac{|\text{End } L_1(\alpha)|}{|\text{Aut } L_1(\alpha)|}$$

3. HALL ALGEBRA ARISING IN A TRIANGULATED CATEGORY

Let \mathcal{C} be a k -additive and \mathcal{C} a (left) locally homological finite triangulated category. For any $X, Y, L \in \mathcal{C}$, by Proposition 2.5', we define

$$\begin{aligned} F_{XY}^L : &= \{X, Y\} \sum_{\alpha \in V(X, Y; L)} \frac{|\text{End} X_1(\alpha)|}{|\text{Aut} X_1(\alpha)|} \\ &= \frac{|\text{Hom}(L, Y)_{X[1]}|}{|\text{Aut}(Y)|\{Y, Y\}} \{L, Y\} = \frac{|\text{Hom}(X, L)_Y|}{|\text{Aut}(X)|\{X, X\}} \{X, L\} \end{aligned}$$

This formula is called Toën's formula ([13, Proposition 5.1]). We will define an associative algebra arising from \mathcal{C} , by using F_{XY}^L as structure constants. For any $X \in \mathcal{C}$, we denote its isomorphism class by $[X]$. Let \mathcal{H} be the \mathbb{Q} -space with the basis $\{u_{[X]} \mid X \in \mathcal{C}\}$. We define

$$u_{[X]} * u_{[Y]} = \sum_{[L]} F_{XY}^L u_{[L]}$$

Since $\text{Hom}(Y, X[1])$ is a finite set, the sum only has finitely many nonzero summands.

Definition 3.1. *Given any objects X, M and L, L' in \mathcal{C} , we set*

$$W(L', L; M \oplus X) = \{((f', -m'), \begin{pmatrix} m \\ f \end{pmatrix}, \theta) \mid ((f', -m'), \begin{pmatrix} m \\ f \end{pmatrix}, \theta) \text{ is a triangle}\}.$$

For fixed $Y, Z \in \mathcal{C}$, we define its subsets

$$W(L', L; M \oplus X)_L^{Y, Z} = \{((f', -m'), \begin{pmatrix} m \\ f \end{pmatrix}, \theta) \mid ((f', -m'), \begin{pmatrix} m \\ f \end{pmatrix}, \theta) \text{ is a triangle},$$

$$\text{Cone}(m) \cong Z[1], \text{ and } \text{Cone}(f) \cong Y\}.$$

and

$$W(L', L; M \oplus X)_{L'}^{Y, Z} = \{((f', -m'), \begin{pmatrix} m \\ f \end{pmatrix}, \theta) \mid ((f', -m'), \begin{pmatrix} m \\ f \end{pmatrix}, \theta) \text{ is a triangle},$$

$$\text{Cone}(m') \cong Z[1], \text{ and } \text{Cone}(f') \cong Y\}.$$

In fact, we have

Proposition 3.2. *The equality holds.*

$$W(L', L; M \oplus X)_L^{Y, Z} = W(L', L; M \oplus X)_{L'}^{Y, Z}$$

The proof of Proposition 3.2 needs the octahedral axiom and pushout property in triangulated category. The following property can be founded in [7].

Proposition 3.3. *The following condition is equivalent to the octahedral axiom: Given a "pushout" square, i.e., a commutative square*

$$(6) \quad \begin{array}{ccc} L' & \xrightarrow{f'} & M \\ \downarrow m' & & \downarrow m \\ X & \xrightarrow{f} & L \end{array}$$

forming a distinguished triangle

$$L' \xrightarrow{\begin{pmatrix} f' & -m' \end{pmatrix}} M \oplus X \xrightarrow{\begin{pmatrix} m \\ f \end{pmatrix}} L \xrightarrow{\theta} L'[1]$$

and θ as above, it can be extended to a commutative diagram

$$(7) \quad \begin{array}{ccccccc} L' & \xrightarrow{f'} & M & \cdots \xrightarrow{g'} & Y & \cdots \xrightarrow{h'} & L'[1] \\ \downarrow m' & & \downarrow m & & \parallel & & \downarrow m'[1] \\ X & \xrightarrow{f} & L & \xrightarrow{g} & Y & \xrightarrow{h} & X[1] \end{array}$$

with $\theta = -gh'$.

Proof of Proposition 3.2. For any $((f', -m'), \begin{pmatrix} m \\ f \end{pmatrix}, \theta) \in W(L', L; M \oplus X)_L^{Y,Z}$, we have the corresponding diagram as follows:

$$(8) \quad \begin{array}{ccccccc} & & Z & & & & \\ & & \downarrow l & & & & \\ L' & \xrightarrow{f'} & M & & & & \\ \downarrow m' & & \downarrow m & & & & \\ X & \xrightarrow{f} & L & \xrightarrow{g} & Y & \xrightarrow{h} & X[1] \\ & & \downarrow n & & & & \\ & & Z[1] & & & & \end{array}$$

Using Proposition 3.3 in horizontal and vertical direction two times, diagram (8) is changed into:

$$(9) \quad \begin{array}{ccccccc} Z & \xlongequal{\quad} & Z & & & & \\ \vdots \downarrow l' & & \downarrow l & & & & \\ L' & \xrightarrow{f'} & M & \cdots \xrightarrow{g'} & Y & \cdots \xrightarrow{h'} & L'[1] \\ \downarrow m' & & \downarrow m & & \parallel & & \downarrow m'[1] \\ X & \xrightarrow{f} & L & \xrightarrow{g} & Y & \xrightarrow{h} & X[1] \\ \vdots \downarrow n' & & \downarrow n & & & & \\ Z[1] & \xlongequal{\quad} & Z[1] & & & & \end{array}$$

This shows $((f', -m'), \begin{pmatrix} m \\ f \end{pmatrix}, \theta) \in W(L', L; M \oplus X)_{L'}^{Y,Z}$, so $W(L', L; M \oplus X)_L^{Y,Z} \subseteq W(L', L; M \oplus X)_{L'}^{Y,Z}$. Similarly, $W(L', L; M \oplus X)_{L'}^{Y,Z} \subseteq W(L', L; M \oplus X)_L^{Y,Z}$. \square

Now we can set

$$W_{Y,Z}(L', L; M \oplus X) := W(L', L; M \oplus X)_{L'}^{Y,Z} = W(L', L; M \oplus X)_L^{Y,Z}$$

Define

$$\begin{aligned} \text{Hom}(M \oplus X, L)_{L'[1]}^{Y,Z[1]} &= \{(m, f) \in \text{Hom}(M \oplus X, L) \mid \\ &\quad \text{Cone}(f) \simeq Y, \text{Cone}(m) \simeq Z[1] \text{ and } \text{Cone}(m, f) \simeq L'[1]\} \end{aligned}$$

and

$$\begin{aligned} \text{Hom}(L', M \oplus X)_L^{Y,Z[1]} &= \{(f', -m') \in \text{Hom}(L', M \oplus X) \mid \\ &\quad \text{Cone}(f') \simeq Y, \text{Cone}(m') \simeq Z[1] \text{ and } \text{Cone}(f', -m') \simeq L\} \end{aligned}$$

The group actions of $\text{Aut } L'$ and $\text{Aut } L$ on $W(L', L; M \oplus X)$ naturally induce the actions on $W_{Y,Z}(L', L; M \oplus X)$. By Proposition 2.6, the orbit spaces under the action of $\text{Aut } L'$ and $\text{Aut } L$ are $\text{Hom}(M \oplus X, L)_{L'[1]}^{Y,Z[1]}$ and $\text{Hom}(L', M \oplus X)_L^{Y,Z[1]}$, respectively. Under the group action of $\text{Aut } L \times \text{Aut } L'$, the orbit space of $W_{Y,Z}(L', L; M \oplus X)$ is denoted by $V_{Y,Z}(L', L; M \oplus X)$. Of course, $V_{Y,Z}(L', L; M \oplus X)$ is a subset of $V(L', L; M \oplus X)$. Naturally, we have the following commutative diagram:

$$\begin{array}{ccc} W_{Y,Z}(L', L; M \oplus X) & \longrightarrow & \text{Hom}(M \oplus X, L)_{L'[1]}^{Y,Z[1]} \\ \downarrow & & \downarrow \\ \text{Hom}(L', M \oplus X)_L^{Y,Z[1]} & \longrightarrow & V_{Y,Z}(L', L; M \oplus X) \end{array}$$

Applying Proposition 2.5', we have

Proposition 3.4. *The equalities hold.*

$$\frac{|\text{Hom}(M \oplus X, L)_{L'[1]}^{Y,Z[1]}|}{|\text{Aut } L|} \frac{|\{M \oplus X, L\}|}{|\{L', L\}\{L, L\}|} = \sum_{\alpha \in V_{Y,Z}(L', L; M \oplus X)} \frac{|\text{End } L_1(\alpha)|}{|\text{Aut } L_1(\alpha)|}.$$

and

$$\frac{|\text{Hom}(L', M \oplus X)_L^{Y,Z[1]}|}{|\text{Aut } L'|} \frac{|\{L', M \oplus X\}|}{|\{L', L\}\{L', L'\}|} = \sum_{\alpha \in V_{Y,Z}(L', L; M \oplus X)} \frac{|\text{End } L_1(\alpha)|}{|\text{Aut } L_1(\alpha)|}.$$

Proposition 3.5. *There exist bijections:*

$$\text{Hom}(X, L)_Y \times \text{Hom}(M, L)_{Z[1]} \rightarrow \bigcup_{[L']} \text{Hom}(M \oplus X, L)_{L'[1]}^{Y,Z[1]}$$

and

$$\text{Hom}(L', X)_{Z[1]} \times \text{Hom}(L', M)_Y \rightarrow \bigcup_{[L]} \text{Hom}(L', M \oplus X)_L^{Y,Z[1]}$$

Proof. The additivity of the Hom functor shows there is an isomorphism

$$\text{Hom}(X, L) \times \text{Hom}(M, L) \cong \text{Hom}(M \oplus X, L)$$

for any $X, M, L \in \mathcal{C}$. This induces a map

$$\text{Hom}(X, L)_Y \times \text{Hom}(M, L)_{Z[1]} \rightarrow \bigcup_{[L']} \text{Hom}(M \oplus X, L)_{L'[1]}^{Y,Z[1]}$$

It is a bijection simply since

$$\text{Hom}(M \oplus X, L)_{L'[1]}^{Y,Z[1]} = (\text{Hom}(X, L)_Y \times \text{Hom}(M, L)_{Z[1]}) \cap \text{Hom}(M \oplus X, L)_{L'[1]}$$

It is similar to prove the second result. \square

Theorem 3.6. *The \mathbb{Q} -space \mathcal{H} is an associative algebra with the \mathbb{Q} -basis $\{u_{[X]} \mid X \in \mathcal{C}\}$, the multiplication:*

$$u_{[X]} * u_{[Y]} = \sum_{[L]} F_{XY}^L u_{[L]}$$

where $F_{XY}^L = \{X, Y\} \cdot \sum_{\alpha \in V(X, Y; L)} \frac{|\text{End}_{X_1}(\alpha)|}{|\text{Aut}_{X_1}(\alpha)|}$ and the unit u_0 .

Proof. For X, Y, Z and $M \in \mathcal{C}$, we need to prove $u_Z * (u_X * u_Y) = (u_Z * u_X) * u_Y$. It is equivalent to prove

$$\sum_{[L]} F_{XY}^L F_{ZL}^M = \sum_{[L']} F_{ZX}^{L'} F_{L'Y}^M.$$

We know that

$$\sum_{[L]} F_{XY}^L F_{ZL}^M = \sum_{[L]} \frac{|\text{Hom}(X, L)_Y|}{|\text{Aut}(X)| \cdot \{X, X\}} \{X, L\} \cdot \frac{|\text{Hom}(M, L)_{Z[1]}|}{|\text{Aut}(L)| \cdot \{L, L\}} \{M, L\}$$

By Proposition 3.5, it equals

$$\frac{1}{|\text{Aut } X| \cdot \{X, X\}} \sum_{[L]} \sum_{[L']} \frac{|\text{Hom}(M \oplus X, L)_{L'[1]}^{Y, Z[1]}|}{|\text{Aut } L|} \cdot \frac{\{M \oplus X, L\}}{\{L, L\}}$$

Dually, the right hand side is

$$\frac{1}{|\text{Aut } X| \cdot \{X, X\}} \sum_{[L']} \sum_{[L]} \frac{|\text{Hom}(L', M \oplus X)_L^{Y, Z[1]}|}{|\text{Aut } L'|} \cdot \frac{\{L', M \oplus X\}}{\{L', L'\}}$$

By Proposition 3.4, they equal to each other. We finish the proof of the theorem. \square

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